NPS55-90-19

NAVAL POSTGRADUATE SCHOOL

Monterey, California

REGRESSION ANALYSIS OF HIERARCHICAL POISSON-LIKE EVENT RATE DATA: SUPER-POPULATION MODEL EFFECT ON PREDICTIONS

Donald P. Gaver
Patricia A. Jacobs
I. G. O'Muircheartaigh

August 1990

Prepared for:
Naval Postgraduate School,
ey, CA 93955

NAVAL POSTGRADUATE SCHOOL,
MONTEREY, CALIFORNIA

Rear Admiral R. W. West, Jr.
Superintendent

Harrison Shull
Provost

This report was prepared in conjunction with research funded under the Naval Postgraduate School Research Council Research Program.

This report was prepared by:

Unclassified
Security Classification of this page

REPORT DOCUMENTATION PAGE

1a Report Security Classification UNCLASSIFIED	1b Restrictive Markings
2a Security Classification Authority	3 Distribution Availability of Report Approved for public release; distribution is unlimited
2b Declassification/Downgrading Schedule	
4 Performing Organization Report Number(s) NPS55-90-19	5 Monitoring Organization Report Number(s)
6a Name of Performing Organization Naval Postgraduate School 6b Office Symbol *(If Applicable)* OR	7a Name of Monitoring Organization
6c Address *(city, state, and ZIP code)* Monterey, CA 93943-5000	7b Address *(city, state, and ZIP code)*
8a Name of Funding/Sponsoring Organization Naval Postgraduate School 8b Office Symbol *(If Applicable)*	9 Procurement Instrument Identification Number O&MN, Direct Funding
8c Address *(city, state, and ZIP code)* Monterey, California Monterey, CA 93943	10 Source of Funding Numbers Program Element Number / Project No / Task No / Work Unit Accession No

11 Title *(Include Security Classification)* Regression Analysis of Hierarchical Poisson-Like Event Rate Data: Superpopulation Model Effect on Predictions

12 Personal Author(s) Gaver, D. P., Jacobs, P. A., and O'Muircheartaigh, I. G.

13a Type of Report	13b Time Covered	14 Date of Report *(year, month,day)*	15 Page Count
Technical	From To	1990, August	

16 Supplementary Notation The views expressed in this paper are those of the author and do not reflect the official policy or position of the Department of Defense or the U.S. Government.

17 Cosati Codes			18 Subject Terms *(continue on reverse if necessary and identify by block number)*
Field	Group	Subgroup	Empirical Bayes prediction; hierarchical models; extra-Poisson variability; Poisson regression; gamma superpopulation; log student-t superpopulation

19 Abstract *(continue on reverse if necessary and identify by block number*

This paper studies *prediction* of future failure (rates) by hierarchical empirical Bayes (EB) Poisson regression methodologies. Both a gamma distributed superpopulation as well as a more robust (long-tailed) log student-t superpopulation are considered. Simulation results are reported concerning predicted Poisson rates. The results tentatively suggest that a hierarchical model with gamma superpopulation can effectively adapt to data coming from a log-Student-t superpopulation particularly if the additional computation involved with estimation for the log-Student-t hierarchical model is burdensome.

20 Distribution/Availability of Abstract [X] unclassified/unlimited [ ] same as report [ ] DTIC users	21 Abstract Security Classification Unclassified
22a Name of Responsible Individual D. P. Gaver	22b Telephone *(Include Area code)* (408) 646-2605 / 22c Office Symbol OR/Gv, OR/Jc

DD FORM 1473, 84 MAR 83 APR edition may be used until exhausted
All other editions are obsolete

security classification of this page
Unclassified

REGRESSION ANALYSIS OF HIERARCHICAL POISSON-LIKE EVENT RATE DATA: SUPERPOPULATION MODEL EFFECT ON PREDICTIONS

D. P. Gaver
P. A. Jacobs
I. G. O'Muircheartaigh

Operations Research Dept.
Naval Postgraduate School
Monterey, California 93943

Operations Research Dept.
Naval Postgraduate School
Monterey, California 93943

Department of Mathematics,
University College Galway,
Ireland

Key Words and Phrases: empirical Bayes prediction; hierarchical models; extra-Poisson variability; Poisson regression; gamma superpopulation; log student-t superpopulation

ABSTRACT

This paper studies *prediction* of future failure (rates) by hierarchical empirical Bayes (EB) Poisson regression methodologies. Both a gamma distributed superpopulation as well as a more robust (long-tailed) log student-t superpopulation are considered. Simulation results are reported concerning predicted Poisson rates. The results tentatively suggest that a hierarchical model with gamma superpopulation can effectively adapt to data coming from a log-Student-t superpopulation particularly if the additional computation involved with estimation for the log-Student-t hierarchical model is burdensome.

1. INTRODUCTION

The following model often provides a useful place from which to commence the analysis of point event process data. First, suppose there is a set of I entities or units, each of which generates an observed history of point events. Take each describing point process to be homogeneous Poisson (λ_i), i = 1, 2, 3, ..., I. The **observed data** appears as (s_i,t_i), s_i being the number of events for process i over active or operating time t_i. Also observed are certain fixed explanatory variable values; x_{ij}, j = 1, 2, ..., p, associated with λ_i. In some literature, e.g., Everitt (1984), such variables are called **manifest**. Second, there is a **latent** quantity, δ_i, associated with λ_i, that is unobservable but influences λ_i behavior. It is convenient to view δ_i, at least provisionally, as being drawn randomly from some superpopulation of values and held fixed thereafter, thus endowing λ_i with its own particular individuality.

We call such a setup **hierarchical**, and ask it to furnish insights and numbers concerning (a) the individual rate values, λ_i, (b) the influence of the explanatory variables upon these rates, and (c) the nature of the superpopulation that gives rise to the latent variable values; future values of the rates, e.g., λ_{I+1}, etc., may be viewed as coming from such a population, at least to a first approximation.

The above model suggests itself for many purposes, one in particular being in risk analysis, e.g., of nuclear power plant safety systems. Such setups are also natural in other reliability-related areas as well, particularly in ones arising in the military. Application may perhaps be made to data reflecting "human unreliability," i.e., the propensity of different individuals to make errors, or experience accidents.

The purpose of this paper is to describe methods for fitting various hierarchical models to the type of data described. Particular attention is devoted to the **prediction problem**: given the past record of an individual item (e.g., human being), how well can one predict its (her) future performance, even if some basic conditions change?

2. THE FORMAL MODEL

The formulation proposed can be written as follows: for i = 1, 2, ..., I, and $\underline{\beta} = (\beta_1, \beta_2, ..., \beta_p)^T$

$$\begin{aligned} \delta_i &\sim \text{IID } g(\cdot;\underline{\theta}) \\ \lambda_i &= f(\underline{x}_i\underline{\beta}, \delta_i), \qquad (2.1) \\ s_i \mid \lambda_i, t_i &\sim \text{Ind. Poiss}(\lambda_i t_i); \end{aligned}$$

$\underline{\theta} = (\theta_1, \theta_2, ..., \theta_r)$ being a parameter identifying g, the density associated with the assumed fixed superpopulation. In what follows we concentrate on certain parametric forms for the link function f and the superpopulation g, and aim at estimating the $\underline{\theta}$-value best representing the superpopulation giving rise to the apparent λ-values. For various reasons, convenience and tradition being influential, we restrict attention to the log-linear model

$$\lambda_i = f(\underline{x}_i\underline{\beta}, \delta_i) = \exp(\underline{x}_i\underline{\beta} + \delta_i). \qquad (2.2)$$

As suggested earlier, the objectives of the analysis will be several-fold, but an important one will be to estimate an individual λ_i-value, i.e., the actual realization of λ_i that prevails. An even more important objective is to predict the number of future events associated with i, $S_i(t)$. This entails finding an estimate $\hat{\underline{\beta}}$ and one for the individualizing parameter δ_i, namely $\hat{\delta}_i$. Estimation will be carried out by assuming that $g(\cdot,\underline{\theta})$, the superpopulation density giving rise to $\boldsymbol{\delta}_i$, is one of a specific parametric family, and first estimating the parameters of that density along with the regression parameters. At a later stage, the estimated parameters $\hat{\underline{\beta}}$ and $\hat{\underline{\theta}}$ are utilized to create estimates of δ_i, and finally λ_i, see Cox and Hinkley (1974), p. 401, Morris (1983), Deely and Lindley (1981), etc. The several-stage or hierarchical analysis is referred to as *parametric empirical Bayes (PEB).*

This work is an extension of Gaver and O'Muircheartaigh (1987) in which discrepancy-tolerant (robust) estimates of δ_i and λ_i were produced and evaluated without consideration of explanatory variables. The major purpose of the present article is to consider the effect of explanatory variables in the context of hierarchical models using quite different models for superpopulations: first, the simple conjugate Gamma, and next the log-Student *t* with a small number of degrees of freedom so that tails are extended, and outliers more apt to be generated.

3. AN EMPIRICAL BAYES APPROACH

The approach taken to providing estimates is traditional; see Berger (1985, Chap. 4). We first remove the condition on $\boldsymbol{\delta}$ for each item to obtain the unconditional likelihood

$$L(\underline{\sigma},\underline{\beta};\underline{s},\underline{t}) = \prod_{i=1}^{I} \int e^{f(\underline{x}_i,\underline{\beta},\delta)t_i} \frac{\left(f(\underline{x}_i,\underline{\beta},\delta)t_i\right)^{s_i}}{s_i!} g(\delta;\underline{\theta})d\delta. \tag{3.1}$$

The latter is then maximized with respect to $\underline{\beta}$ and $\underline{\theta}$ to produce $\hat{\underline{\beta}}$ and $\hat{\underline{\theta}}$. These quantities are then inserted into the expression for the posterior density of $\boldsymbol{\delta}$,

$$g_p(\delta) \equiv g_p(\delta;\underline{\theta},s_i,t_i,\underline{x}_i\underline{\beta}) = K_i e^{-f(\underline{x}_i,\hat{\underline{\beta}},\delta)t_i}\left(f(\underline{x}_i\hat{\underline{\beta}},\delta)t_i\right)^{s_i} g(\delta;\hat{\theta})$$

where the constant K_i is a normalizing factor. A point estimator of λ_i is taken to be the posterior mean (other options of course available),

$$\hat{\lambda}_i = \int f(\tilde{\underline{x}}_i\hat{\underline{\beta}},\delta)g_p(\delta)d\delta \tag{3.2}$$

where $\tilde{\underline{x}}_i$ is the value of the explanatory variable for conditions anticipated when the estimate is to be applied; if $\tilde{\underline{x}}_i = \underline{x}_i$ then we have $\hat{\lambda}_i$, an empirical Bayes estimate of λ_i, for conditions under which the data were taken; this will often be a shrunken estimator that has a smaller mean-squared error than does a simple individual estimator. If $\tilde{\underline{x}}_i$ refers to other (e.g., future) conditions then $\hat{\lambda}_i$ calculated by (3.2) may be called the *mean predictive rate.* If more information is desired then the entire predictive distribution is needed:

$$\tilde{p}(\tilde{s}_i) = \int e^{-f(\tilde{\underline{x}}_i,\hat{\underline{\beta}},\delta)\tilde{t}_i} \left(f\left(\tilde{\underline{x}}_i,\hat{\underline{\beta}},\delta\right)\tilde{t}_i\right)^{\tilde{s}_i} \frac{1}{\tilde{s}_i!} g_p(\delta) d\delta, \tag{3.3}$$

this approximates the conditional probability of $\tilde{s}_i$ future events for item i, given that it is exposed for time $\tilde{t}_i$ and under conditions $\tilde{\underline{x}}_i$.

It is apparent that the approximation so obtained may be under-variable, in that it treats $\hat{\underline{\beta}}$ and $\hat{\underline{\theta}}$ as fixed and known in (3.2) and (3.3). The **hierarchical Bayes analysis** described by Berger (1985, Chap. 4) is a substitute that avoids that criticism. This defect is undeniable, but some appreciation for the magnitude of the effect can be obtained by bootstrapping. Of more concern to us has been investigation of the effect of superpopulation *model choice*: how different can actual *prediction* be in simple situations modeled quite differently? We proceed to compare and contrast two models, one conjugate Gamma and the other longer-tailed and hence outlier-prone.

4. GAMMA LATENT VARIABLE POISSON REGRESSION (GALVPR).

It is conventional and convenient to invoke the gamma density to represent the random effect in (2.1); see Lawless (1987 a,b) and Anscombe (1950) for examples. Thus

$$g(u;a) = e^{-u/a} \frac{(u/a)^{a^{-1}-1}}{\Gamma\left(a^{-1}\right)} \frac{1}{a} \tag{4.1}$$

is the superpopulation model, from which

$$\begin{aligned} E[\delta] &= 1 \\ Var[\delta] &= a. \end{aligned} \tag{4.2}$$

Lawless (1987b) gives expressions for the ln-likelihood and its derivatives for this hierarchical model. It turns out, however, that a more satisfying parameterization is in terms of $\theta = \ln a$ when the mle stage is undertaken. Since a one-parameter gamma density is used, the regression has a constant term; that is $x_{i1} = 1$. For convenience we provide expressions for the ln-likelihood and its derivatives using our parameterization.

In the present parameterization, then,

$$\lambda = U \exp[\underline{x}_i \underline{\beta}] \tag{4.3}$$

so $\exp(\delta) = U$ is gamma. In order to form the likelihood element in (3.1) it is only necessary to integrate to obtain the explicit form

$$L(\underline{\theta}, \underline{\beta}; \underline{s}, \underline{t}) = \prod_{i=1}^{I} \frac{\Gamma(s_i + e^{-\theta})}{s_i! \Gamma(e^{-\theta})} \left(\frac{e^{\theta} c_i t_i}{e^{\theta} c_i t_i + 1} \right)^{s_i} \left(\frac{1}{e^{\theta} c_i t_i + 1} \right)^{e^{\theta}} \tag{4.4}$$

where $c_i = \exp\{\underline{x}_i \underline{\beta}\}$. The log-likelihood is

$$l(\underline{\theta}, \underline{\beta}; \underline{s}, \underline{t}) = \sum_{i=1}^{I} \left\{ \left(\sum_{j=0}^{s_i - 1} \ln(1 + j e^{\theta}) \right) - \ln s_i! + s_i \left[\ln c_i t_i - \ln(e^{\theta} c_i t_i + 1) \right] - e^{-\theta} \ln(e^{\theta} c_i t_i + 1) \right\}$$

(4.5)

where $\sum_{j=0}^{s_i - 1} \ln(1 + j e^{\theta}) = 1$ if $s_i = 0$.

The following derivatives can be obtained.

$$\frac{\partial l}{\partial \theta} = e^{\theta} \sum_{i=1}^{I} \left\{ \left(\sum_{j=0}^{s_i - 1} \frac{j}{1 + j e^{\theta}} \right) + e^{-2\theta} \ln[e^{\theta} c_i t_i + 1] - \frac{c_i t_i}{e^{\theta} c_i t_i + 1} [s_i + e^{-\theta}] \right\} \tag{4.6}$$

$$\frac{\partial^2 l}{\partial \theta^2} = \frac{\partial l}{\partial \theta} + e^{2\theta} \left\{ \sum_{i=1}^{I} \left\{ \left(\sum_{j=0}^{s_i - 1} - \left(\frac{j}{1 + j e^{\theta}} \right)^2 \right) - 2 e^{-3\theta} \ln[e^{\theta} c_i t_i + 1] + 2 \frac{c_i t_i e^{-2\theta}}{e^{\theta} c_i t_i + 1} + \left(\frac{c_i t_i}{e^{\theta} c_i t_i + 1} \right)^2 [s_i + e^{-\theta}] \right\} \right\}$$

(4.7)

$$\frac{\partial l}{\partial \beta_k} = \sum_{i=1}^{I} \left[\frac{s_i - c_i t_i}{e^{\theta} c_i t_i + 1} \right] x_{ik} \tag{4.8}$$

$$\frac{\partial^2 l}{\partial\beta_k\partial\beta_j} = -\sum_{i-1}^{I}\frac{\left[s_i e^{\theta}c_i t_i + c_i t_i\right]}{\left(e^{\theta}c_i t_i + 1\right)^2}x_{ij}x_{ik} \tag{4.9}$$

where the summations involving s_i-1 are set equal to zero when s_i=0. Further,

$$E\left[\frac{-\partial^2 l}{\partial\beta_k\partial\beta_j}\right] = \sum_{i=1}^{I}\left(\frac{c_i t_i}{e^{\theta}c_i t_i + 1}\right)x_{ij}x_{ik}. \tag{4.10}$$

A Newton-like iterative procedure is used to solve the system of equations

$$0 = \frac{\partial l}{\partial\theta} \tag{4.11}$$

$$0 = \frac{\partial l}{\partial\beta_k}. \tag{4.12}$$

If s_i is large then evaluating the sums appearing in (4.5), (4.6), (4.7), (4.13) and elsewhere tends to be time-consuming. However all such sums are well-behaved (of monotonic formations) and can be well-approximated by integrals. This feature is not, but easily can be, included in our programs.

If $\{\beta_k\}$ were known, then a Newton procedure to estimate θ would be to recursively solve the linear equation

$$0 = \frac{\partial l}{\partial\theta} = \frac{\partial l}{\partial\theta}\bigg|_{\theta=\theta^0} + \left(\frac{\partial^2 l}{\partial\theta^2}\bigg|_{\theta=\theta^0}\right)\left(\theta-\theta^0\right) \tag{4.12}$$

where θ^0 is a current estimate of θ. Note that if $\frac{\partial l}{\partial\theta} = 0$, then

$$\frac{\partial^2 l}{\partial\theta^2} = g(\theta) = e^{2\theta}\left\{\sum_{i=1}^{I}\left\{\left(\sum_{j=0}^{s_i-1} -\left(\frac{j}{1+je^{\theta}}\right)^2\right) - 2e^{-3\theta}\ln\left[e^{\theta}c_i t_i + 1\right] + \frac{2c_i t_i e^{-2\theta}}{e^{\theta}c_i t_i + 1} + \left(\frac{c_i t_i}{e^{\theta}c_i t_i + 1}\right)^2\left[s_i + e^{-\theta}\right]\right\}\right\}. \tag{4.13}$$

Hence, (4.12) can be rewritten as

$$\theta-\theta^0=\left[\sum_{i=1}^{I}\left\{\left(\sum_{j=0}^{s_i-1}\frac{j}{1+je^{\theta}}\right)+e^{-2\theta}\ln\left[e^{\theta}c_it_i+1\right]-\frac{c_it_i}{e^{\theta}c_it_i+1}\left[s_i+e^{\theta}\right]\right\}\right]$$
$$\times\left[e^{\theta}\left\{\sum_{j=0}^{I}\left\{\left(\sum_{j=0}^{s_i-1}\left(\frac{j}{1+je^{\theta}}\right)^2\right)+2e^{-3\theta}\ln\left[e^{\theta}c_it_i+1\right]-\frac{2c_it_ie^{-2\theta}}{e^{\theta}c_it_i+1}-\left(\frac{c_it_i}{e^{\theta}c_it_i+1}\right)^2\left[s_i+e^{-\theta}\right]\right\}\right\}\right]^{-1}. \tag{4.14}$$

To obtain an initial estimate of θ, note that, letting $N_i(t_i)$ denote the i^{th} random variable of the number of observed events,

$$E\left[N_i(t_i)\right]=c_it_i\,; \tag{4.15}$$

$$Var\left[N_i(t_i)\right]=c_it_i\left[1+c_it_ie^{\theta}\right]. \tag{4.16}$$

Thus, $\left[(N_i(t_i)-c_it_i)/\sqrt{c_it_i}\right]$ has mean 0 and variance $[1+c_it_ie^{\theta}]$. We propose starting the iterative procedure to find θ by computing

$$\hat{m}=\frac{1}{I}\sum_{i=1}^{I}\left(\frac{s_i-c_it_i}{\sqrt{c_it_i}}\right)^2. \tag{4.17}$$

If $\hat{m}\le 1$, then a log-linear model is used to describe the data. If $\hat{m}>1$, then the initial estimate of θ is

$$\hat{\theta}_0=\ln\left[(\hat{m}-1)\left[\frac{1}{I}\sum_{i=1}^{I}c_it_i\right]^{-1}\right]. \tag{4.18}$$

If θ were known, then $\{\beta_k\}$ could be estimated with generalized linear model software in the following manner, (cf. McCullagh and Nelder [1983]).

A Newton iteration to solve the equations $0=\dfrac{\partial l}{\partial\beta_k}$ is to solve the system of equations

$$0=\left(\frac{\partial l}{\partial\beta_k}\Big|\ \underline{\beta}=\underline{\beta}^0\right)+\sum_{j=1}^{p}\left(E\left[\frac{\partial^2 l}{\partial\beta_k\partial\beta_j}\right]\Big|\ \left(\underline{\beta}=\underline{\beta}^0\right)\right)\left(\beta_j-\beta_j^0\right) \tag{4.19}$$

where $\underline{\beta}^0$ is the current estimate of $\underline{\beta}$.

Put

$$w_i = \left[\frac{c_i t_i}{e^{\theta} c_i t_i + 1} \right]^{\frac{1}{2}} \tag{4.20}$$

where $c_i = \exp\{\underline{x}_i \underline{\beta}^0\}$.

Equation (4.19) can be rewritten as

$$0 = \sum_{i=1}^{I} \left(y_i - \sum_{j=1}^{p} u_{ij} \beta_j \right) u_{ik} \tag{4.21}$$

k=1, ..., p

where

$$y_i = [s_i - c_i t_i][c_i t_i (e^{\theta} c_i t_i + 1)]^{-\frac{1}{2}} + \sum_{j=1}^{p} u_{ij} \beta_j^0 \tag{4.22}$$

and

$$u_{ij} = w_i x_{ij}. \tag{4.23}$$

The equations of (4.21) are the normal equations for a least squares regression.

The following is an iterative procedure to obtain estimates of θ and $\{\beta_k\}$

a) Fit a log-linear model stopping after one iteration

1. Start with

$$\underline{x}_i \underline{\beta}^0 = \ln\left[\left(s_i + \frac{1}{2} \right) / t_i \right]$$

2. Solve the equation (4.21)

with

$$w_i = \left[s_i + \frac{1}{2} \right]^{\frac{1}{2}};$$

$$u_{ij} = w_i x_{ij};$$

$$y_i = -w_i \left(\frac{s_i}{w_i} \right) + w_i \underline{x}_i \underline{\beta}^0.$$

b) Find the initial estimate of θ by evaluating (4.18). If $m \leq 1$, use the log-linear model of a) to describe the data.

I. Next estimate $\{\beta_k\}$: Evaluate and solve equations (4.20) - (4.23).

II. Next estimate θ: Evaluate and solve equation (4.14).

III. Continue alternating between I and II until convergence.

In the simulation experiments described in Section 7 the above-obtained estimate of θ occasionally either cycled among negative values or became very large and positive. In these cases II was replaced by a search of the marginal likelihood for θ with fixed $\{\beta_k\}$.

5. ROBUST HIERARCHICAL POISSON REGRESSION (ROLVPR): THE LOG-STUDENT *t* SUPERPOPULATION

As an alternative to the GALVPR model, allow $\boldsymbol{\delta}$ to have the Student t density

$$g(\delta;\tau^2,d)=\frac{C(d)}{\left(1+\delta/\tau^2\right)^{\frac{d+1}{2}}}; \tag{5.1}$$

this distribution is adjustably longer-tailed than is the log-Gamma distribution (for $\boldsymbol{\delta}$) of the previous model, and hence better represents outliers and extreme extra-Poisson variability. The parameter d is the "degrees of freedom" for the Student t; for the present purpose a low value of d (e.g., d = 3-5) is useful. The Student-t model for log failure rate was introduced in Gaver and O'Muircheartaigh (1987). There it was pointed out that the marginalization step of (3.1) could be performed using Gauss-Hermite numerical integration; see Naylor and Smith (1982). In this paper we employ a variant of the Gauss-Hermite technique that involves an initial correction by Laplace's method.

The procedure currently adopted for fitting the regression parameters $\underline{\beta}$ in addition to the Student t parameter τ proceeds iteratively: first explain as much item-to-item variability as possible by suitably weighted regression, then alter the model to approximately adjust for regression effects and apply the methodology of Gaver and O'Muircheartaigh (1987) to estimate τ^2. This value then provides refined weights for a new regression. We speak of **rocking** back and forth between the regression and latent variable stages.

5.1. Rocking Algorithm when δ_i ~ Student (μ, τ, d)

Here is how the above procedure operates when latent variables are Student t so as to represent adjustably long-tailed outlier-prone regressions; d $\geq$ 1 is a tuning parameter with $\mathrm{Var}[\boldsymbol{\delta}] = \tau^2 d/(d-2)$ if d > 2.

a) Regress $y_i(1) = \sqrt{s_i}\ \ln(s_i/t_i)$ on $\sqrt{s_i}\ \underline{x}_i$; (5.1)

Replace $s_i/t_i = 0/t_i$ by $1/3t_i$. Obtain $\hat{\underline{\beta}}(1)$.

b) In the ith likelihood component obtained by integrating out with respect to the δ_i-distribution,

$$L_i\left(\tau^2,\underline{\beta},s_i,t_i\right)=\int_{-\infty}^{\infty}e^{-\lambda_i(z)t_i}\left(\lambda_i(z)\right)^{s_i}\frac{C(d)}{\left[1+\left(z^2/\tau^2 d\right)\right]^{\frac{d+1}{2}}}\frac{1}{\tau}dz \tag{5.2}$$

where

$$\ln\lambda_i(z)=\underline{x}_i\underline{\beta}+z, \tag{5.3}$$

replace t_i by $t_i e^{\underline{x}_i\hat{\beta}(1)}=t_i(1)$. Now numerically optimize (5.2) by choice of $\tau=\hat{\tau}(2)$; $\hat{\tau}(1)$ is a moment estimator. Details of the likelihood integral approximation and optimizations are furnished in Section 6.

c) Regress $y_i(2)=\left(\frac{1}{s_i}+\hat{\tau}^2(2)\frac{d}{d-2}\right)^{\frac{-1}{2}}$ ln (s_i/t_i) on $\left(\frac{1}{s_i}+\hat{\tau}^2(2)\frac{d}{d-2}\right)^{\frac{-1}{2}}\underline{x}_i$

where t_i and s_i are the original data values. Obtain $\hat{\beta}(2)$.

d) In step b above replace t_i by $t_i e^{x_i\hat{\underline{\beta}}(2)}$ and $\underline{x}_i\underline{\beta}$ by $\underline{x}_i\underline{\beta}(2)$ and again numerically optimize to find $\hat{\tau}(3)$.

e) Return to step c) with $\hat{\tau}^2(3)$.

f) Continue to convergence of $\left\{\hat{\underline{\beta}}(k)\right\},\left\{\tau^2(k)\right\}$.

The above procedure converges rapidly in our experience, giving results in close agreement with the simultaneous optimization of the likelihood with respect to τ and β. The latter is a much more computationally demanding procedure than is rocking.

6. LIKELIHOOD COMPUTATION

An essential part of the preceding algorithm is the numerical evaluation of likelihoods of this form:

$$L\left(\tau^2;(\underline{s},\underline{t})\right)=\prod_{i=1}^{I}L_i\left(\tau^2;s_i,t_i\right), \tag{6.1}$$

where

$$L_i\left(\tau^2;s_i,t_i\right)=\int_{-\infty}^{\infty}e^{-\lambda_i(z)t_i}\frac{[\lambda_i(z)]^{s_i}}{s_i!}\frac{C(d)}{\left[1+z^2/\tau^2 d\right]^{(d+1)/2}}\frac{1}{\tau}dz$$

$$\equiv\int_{-\infty}^{\infty}e^{-Q_i(z)}dz. \tag{6.2}$$

Under the log-linear model

$$\ln\lambda_i(z)=\underline{x}_i\underline{\beta}+z \tag{6.3}$$

so

$$Q_i(z)=\lambda_i(z)t_i-s_i\ln\lambda_i(z)+\frac{d+1}{2}\ln\left[1+z^2/\tau^2 d\right]+\ln\tau, \tag{6.4}$$

omitting irrelevant constants. In order to evaluate the integral in (6.2) approximately but reasonably accurately we apply either (a) a version of Laplace's method, in which $Q_i(z)$ is approximated by a quadratic and integrated explicitly; alternatively (b) apply a refined version of (a) involving Gauss-Hermite integration of the error resulting from the quadratic approximation to (6.3). Here is a sketch of the process. In what follows we will modify the time to be $t_i e^{\underline{x}_i\underline{\beta}}$.

6.1. Laplace Method, and a Refinement.

To compute $L_i=\int_{-\infty}^{\infty}e^{-Q_i(z)}dz$, the ith likelihood component, we begin by approximating Q_i by a quadratic as follows. Since $\lambda_i = e^z$, t_i is modified as indicated above, and

$$-Q_i(z)=-\lambda_i t_i+s_i\ln\lambda_i-\frac{d+1}{2}\ln\left(1+z^2/\tau^2 d\right)-\ln\tau,$$

$$-\frac{dQ_i}{dz}=-e^z t_i+s_i-\left(\frac{d+1}{d}\right)\frac{1}{\left[1+z^2/\tau^2 d\right]}\frac{z}{\tau^2} \tag{6.5}$$

$$\equiv -e^z t_i+s_i-w(z)z/\tau^2$$

where

$$w(z)=\frac{d+1}{d}\left[1+z^2/\tau^2 d\right]$$

is considered to be a weight. Now $-dQ_i/dz=0$ entails the equation

$$e^{z} = \frac{1}{t_i}\left[s_i + \frac{z}{\tau^2}\left(\frac{\frac{d+1}{d}}{\left[1+z^2/\tau^2 d\right]}\right)\right]. \tag{6.6}$$

Equation (6.6) may have two solutions. We obtain a single reasonable approximate solution to (6.6) as follows: An initial solution to (6.6) is

$$\begin{aligned} z_i(0) &= \ln(s_i / t_i) \text{ if } s_i > 0, \\ z_i(0) &= \ln(1/3t_i) \text{ if } s_i = 0; \end{aligned} \tag{6.7}$$

other replacements for the zero count situation are possible.
Let $\bar{z}_i$ be the solution to the equation

$$e^{z} = \frac{1}{t_i}\left[s_i - zw(z_i(0))/\tau^2\right]$$

after one Newton-Raphson iteration starting at $z_i(0)$.

Next evaluate an approximation to $Q_i''(\bar{z}_i)$. First approximate

$$Q_i'(z) \approx e^{z}t_i - s_i + w(\bar{z}_i)\frac{z}{\tau^2}. \tag{6.8}$$

Hence,

$$Q_i''(z) \approx e^{z}t_i + w(\bar{z}_i)\frac{1}{\tau^2}. \tag{6.9}$$

Finally approximate $e^{\bar{z}_i}$ by s_i/t_i resulting in the approximation

$$Q_i''(\bar{z}_i) \approx s_i + w_i(\bar{z}_i)\frac{1}{\tau^2}. \tag{6.10}$$

Write

$$Q_i(z) = Q_i(\bar{z}_i) + (z-\bar{z}_i)^2 \frac{1}{2} Q_i''(\bar{z}_i) + R_i(z); \tag{6.11}$$

Laplace's method assumes that $R_i(z)$ is negligible and hence

$$
\begin{aligned}
L_i\left(\tau^2;s_i,t_i\right) &= \int_{-\infty}^{\infty} e^{-Q_i(z)}dz \\
&\cong e^{-Q(\bar{z}_i)}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\bar{z}_i)^2 Q_i''(\bar{z}_i)}dz \qquad (6.12)\\
&= e^{-Q(\bar{z}_i)}\sqrt{2\pi}\,/\,\sqrt{Q_i''(\bar{z}_i)}
\end{aligned}
$$

so the log-likelihood

$$
\ell\left(\tau^2;\underline{s},\underline{t}\right) \cong -\sum_{i=1}^{I}\left[Q_i(\bar{z}_i)+\frac{1}{2}\ln Q_i''(\bar{z}_i)\right] \qquad (6.13)
$$

which can be numerically optimized by choice of τ^2 for fixed tuning constant value d (in principle optimization on d can also be included).

Improved numerical results have been achieved by writing

$$
\begin{aligned}
L_i\left(\tau^2;s_i,t_i\right) &= \int_{-\infty}^{\infty} e^{-Q_i(z)}dz \qquad (6.14)\\
&= e^{-Q(\bar{z}_i)}\sqrt{2\,/\,Q_i''(\bar{z}_i)}\int_{-\infty}^{\infty} e^{-w^2}e^{-R_i^{*}(w)}dw
\end{aligned}
$$

with

$$
R_i^{*}(w) = R_i\left(\sqrt{2\,/\,Q_i''(\bar{z}_i)}\,w+\bar{z}_i\right), \qquad (6.15)
$$

$R_i(w)$ being defined by (6.11). The integration is then performed by Gauss-Hermite technique, i.e., by replacing the integral by a finite sum at points w_i determined by zeros of the Hermite polynomials; see Abramowitz and Stegun (1964). Experience has shown that the above produce numerical results that agree well with other numerical methods such as that of Naylor and Smith (1982); however, the unadorned Laplace, (6.12), may sometimes be satisfactory, and is certainly more quickly computed, which is a virtue if bootstrapping is undertaken.

Alternative computational procedures exist and have virtues. The Newton-like iteration applied to the Gamma model of Section 4 can be adapted to the log-Student model, but we have not undertaken this as yet. A sampling-based approach of Gelfand and Smith (1988) is a natural option, but at present appears unnecessarily computer-intensive. As will appear, even the apparently crude rocking approximation proposed leads to interesting contrasts between predictions made by the conventional conjugate Gamma and the robustifying Student.

7. NUMERICAL ILLUSTRATIONS

In order to illustrate the performance of the two proposed prediction schemes we have performed extensive simulations. These illustrate the anticipated comparative performance of GALVPR and ROLVPR: the latter is often better able to adapt to the appearance of large outlier rates by refusing to shrink them down as extensively as do the former. The difference between the predictions made by the two schemes is less noticeable for small rates; here the behavior of the gamma-based approach, GALVPR, may actually be superior, probably because of the approximations made when implementing the Student ROLVPR model. Improvements in the current procedure for fitting the latter, e.g., when counts are zero, are likely to show up as reduced upward shrinkages.

Simulation Experiment

The present simulations are all based on a group of I=20 items. For the log-linear rate of (2.2) $\underline{x}_i\underline{\beta} = \beta_1 + \beta_2 x_i$, and $x_i = +1$ for i = 1, 2, ..., 10, $x_i = -1$ for i = 11, 12, ..., 20. In addition β_1=0.5 and β_2 = 0.1 and 0.3, while $t_i = 2$ throughout. For each experiment 20 Poisson rates were then generated from the Student model with $\tau = 1.0$ and d = 5, and for each rate a single Poisson data point was generated with mean $\lambda_i t_i$. These then constitute the observed counts from which predictions are made. Each prediction is viewed as a point estimate of the underlying Poisson mean giving rise to the corresponding observed count; it is a natural point estimate for a future count. The predictions are chosen to be the means of the posterior distributions from the GALVPR and ROLVPR model specifications, where each model is fitted to the data (20 counts, plus values of x_i) for the particular experiment, meaning that β_i, i = 1, 2, and a, for GALVPR, and τ^2, ROLVPR were estimated as described earlier. These models were actually fitted by two methods: (a) to all count data in the experiment, including that for the item whose rate is predicted, and (b) to all data, but omitting the observation for the item to be predicted, i.e., in *cross-validation mode*.

An illustration of a particular experimental outcome, and the corresponding predictions appears in Table I. Note that for this particular data set the average mean square error of ROLVPR no-cross-validation predictions is the smallest. This is not always so; see the figures for comparisons of mean-squared-errors for the two shrunken predictions, and raw predictions.

TABLE I
SAMPLE COMPARISON OF RATES AND ESTIMATES
$\beta_1 = 0.5, \beta_2 = 0.1, \tau = 1, d = 5$

Number	Co-variate	Ob-served	True	Raw	No Cross-validation		Cross-validation	
i	x_i	s_i	λ_i	λ_i(raw)	λ_i(GA)	λ_i(Stu.t)	λ_i(GA)	λ_i(Stu.t)
1	+1	152	87.57	76.00	74.56	76.28	53.56	73.68
2	+1	3	1.44	1.50	1.66	1.62	1.66	1.66
3	+1	2	0.47	1.00	1.17	1.23	1.17	1.26
4	+1	2	3.51	1.00	1.17	1.23	1.17	1.26
5	+1	0	0.65	0.00	0.19	0.47	0.21	0.52
6	+1	0	0.16	0.00	0.19	0.47	0.21	0.52
7	+1	5	2.07	2.50	2.64	2.45	2.63	2.48
8	+1	3	2.07	1.50	1.66	1.62	1.66	1.66
9	+1	8	1.20	4.00	4.10	3.76	4.10	3.77
10	+1	1	1.11	0.50	0.68	0.85	0.68	0.86
11	-1	9	2.87	4.50	4.31	4.15	4.28	4.16
12	-1	2	0.80	1.00	1.10	1.19	1.10	1.19
13	-1	3	0.43	1.50	1.56	1.59	1.56	1.57
14	-1	0	1.22	0.00	0.18	0.46	0.19	0.52
15	-1	9	5.44	4.50	4.31	4.15	4.28	4.16
16	-1	7	3.18	3.50	3.40	3.26	3.38	3.27
17	-1	11	5.40	5.50	5.23	5.10	5.16	5.08
18	-1	0	0.40	0.00	0.18	0.46	0.19	0.52
19	-1	3	0.66	1.50	1.56	1.59	1.56	1.57
20	-1	0	0.03	0.00	0.18	0.46	0.19	0.52
			MSE's:	7.83	9.16	7.34	58.94	10.61

TABLE II
SAMPLE COMPARISON OF RATES AND ESTIMATES
$\beta_1 = 0.5, \beta_2 = 0.1, \tau = 1, d = 5$

Observed	True	Raw	No Cross-validation		Cross-validation	
s_i	λ_i	λ_i(raw)	λ_i(GA)	λ_i(Stu.t)	λ_i(GA)	λ_i(Stu.t)
267	138.43	133.50	132.11	137.56	101.48	130.66
0	0.01	0.00	0.17	0.48	0.19	0.51
3	1.35	1.50	1.65	1.64	1.65	1.65
12	6.55	6.00	6.10	5.62	6.10	5.59
0	0.29	0.00	0.17	0.48	0.19	0.51
2	2.21	1.00	1.16	1.25	1.16	1.25
2	1.32	1.00	1.16	1.25	1.16	1.25
5	3.47	2.50	2.64	2.46	2.64	2.48
0	0.57	0.00	0.17	0.48	0.19	0.51
2	0.98	1.00	1.16	1.25	1.16	1.23
7	2.18	3.50	3.52	3.42	3.52	3.39
3	0.47	1.50	1.60	1.71	1.60	1.71
36	20.25	18.00	17.41	17.48	16.91	17.40
4	1.68	1.00	1.12	1.31	1.12	1.32
2	3.72	2.00	2.08	2.12	2.08	2.12
0	0.39	0.00	0.17	0.51	0.18	0.58
6	2.28	3.00	3.04	3.97	3.04	2.96
2	1.48	1.00	1.12	1.31	1.12	1.32
9	2.40	4.50	4.58	4.30	4.48	4.29
10	4.46	5.00	4.96	4.76	4.95	4.77
	MSE's	2.22	2.90	1.08	69.51	4.09

NOTE: this is an independent experiment from the same setup as that of Table 1.

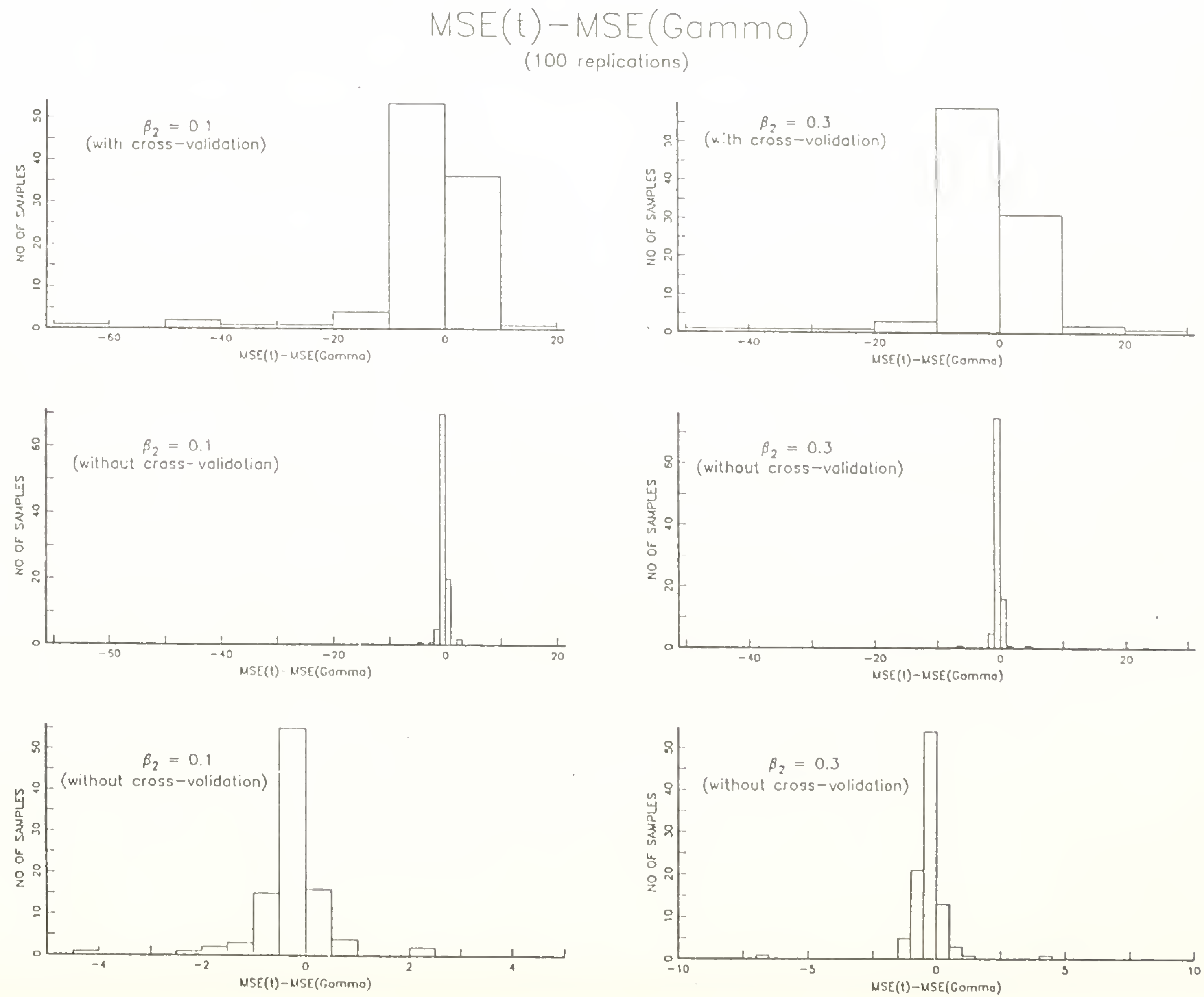
MSE(t)-MSE(Gamma)
(100 replications)
β_2 = 0.1
(with cross-validation)
β_2 = 0.3
(with cross-validation)
β_2 = 0.1
(without cross-validation)
β_2 = 0.3
(without cross-validation)
β_2 = 0.1
(without cross-validation)
β_2 = 0.3
(without cross-validation)
NO OF SAMPLES
MSE(t)-MSE(Gamma)

Figure 1

MSE(Shrunken estimator) − MSE(Raw rate)

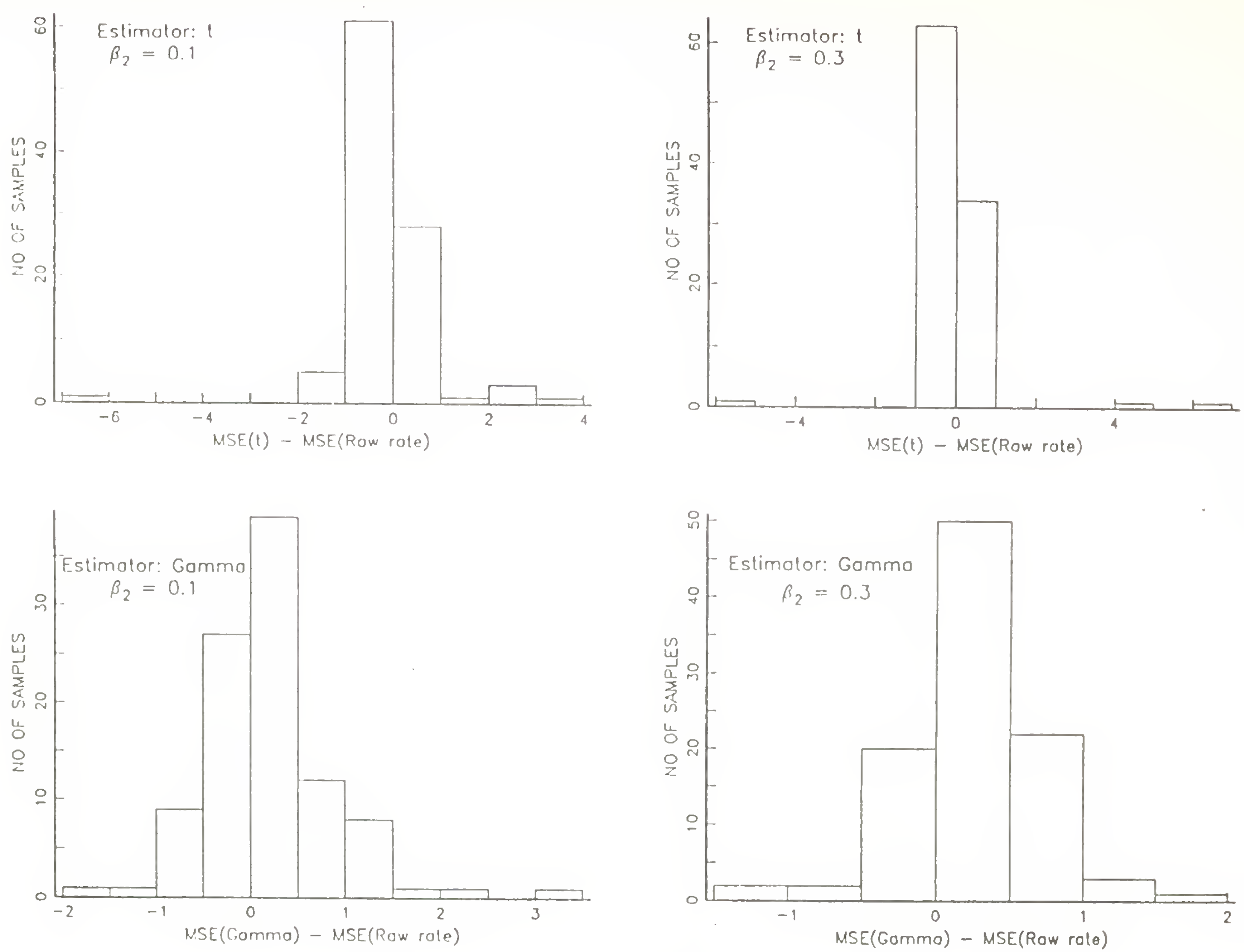

Figure 2

Table II provides another illustration of the estimates' performance, this time with fewer extreme outliers. Figure 1 exhibits histographs of the mean-squared error difference, MSE(ROLVPR)-MSE(GALVPR), for 100 replications of the above specific simulation. Note that the advantage is in favor of ROLVPR in the majority of the experiments, with an exceptional advantage displayed in some cases. Often it is in a few cases of exceptionally large rates, and counts, that ROLVPR excels. Figure 2 compares the mean-squared errors of each shrunken no-cross validated estimator with the corresponding mean square error of the raw-rate estimators; the raw rate estimator is simply the count divided by $t_i = 2$. For these data sets the indications are that ROLVPR improves upon RAW most of the time when $\beta_2=0.1$ and when $\beta_2=0.3$ (although less decisively), while RAW improves upon GALVPR most of the time; neither victory is decisive. These results are perhaps not surprising when one refers to Morris (1983), Theorem 1 and subsequent discussion. It appears that the convenient conjugate can adapt to non-gamma data quite well in many of the present cases at least.

An undoubted disadvantage of the ROLVPR procedure is its computer intensivity: computation of its estimators requires far more time than does GALVPR because a root must be found, (6.6), and a numerical integration performed. Search is on for a more tractable representation of a "robust g" that permits analytical rather than computational evaluation. The inverse Gaussian is a candidate; see Dean et al., (1989). Conceivably such an adoption will result in better results for small-rate situations. Needless to say RAW, which quotes $\lambda_i(RAW) = s_i/t_i$ is by far the most economical. Of course it may not be used if the covariate value, x_i, changes.

REFERENCES

Abramowitz, M. and Stegun, I. A., (1964), *Handbook of Mathematical Functions*, U. S. Department of Commerce, National Bureau of Standards, Applied Mathematic Series 55, Seventh Printing, 1968.

Anscombe, F. J. (1950). Sampling theory of the negative binomial and logarithmic series distributions. *Biometrika* **37** 358-382.

Berger, J. O., (1985), *Statistical Decision Theory and Bayesian Analysis*, Second Edition, New York: Springer-Verlag.

Cox, D. R., and Hinkley, D. V., (1974), *Theoretical Statistics*, London: Chapman and Hall.

Dean, C., Lawless, J. F. and Willmot, G. E. (1989), "A mixed Poisson-inverse-Gaussian regression model," *The Canadian Journal of Statistics*, **17**, 171-181.

Deely, J. J., and Lindley, D. V., (1981), "Bayes empirical Bayes," *Journal of the American Statistical Association*, **76**, 833-841.

Everitt, B. S., (1984), *An Introduction to Latent Variable Models*, London: Chapman and Hall.

Gaver, D. P., and O'Muircheartaigh, I. G., (1987), "Robust empirical Bayes analysis of event rates," *Technometrics* **29**, 1-15.

Gelfand, A. E., and Smith, A. F. M., (1988), "Sampling based approaches to calculating marginal densities," Technical Report, Department of Mathematics, Nottingham Statistics Group, University of Nottingham.

Lawless, J. F., (1987a), "Regression methods for Poisson process data," *Journal of the American Statistical Association*, **82**, pp. 808-815.

Lawless, J. F., (1987b), Negative binomial regression models. *Canad. J. Statist.* **15**, 209-226.

McCullagh, P., and Nelder, J. A., (1983), *Generalized Linear Models*, London: Chapman and Hall.

Morris, C. (1983), "Parametric empirical Bayes inference: theory and applications," (with discussion), *Journal of the American Statistical Association*, **78**, 47-65.

Naylor, J. C., and Smith, A. F. M., (1982), "Applications of a method for the efficient computation of posterior distributions," *Applied Statistics* **31**, 214-225.

INITIAL DISTRIBUTION LIST

1. Library (Code 0142)......2
Naval Postgraduate School
Monterey, CA 93943-5000

2. Defense Technical Information Center......2
Cameron Station
Alexandria, VA 22314

3. Office of Research Administration (Code 012)......1
Naval Postgraduate School
Monterey, CA 93943-5000

4. Prof. Peter Purdue......1
Code OR-Pd
Naval Postgraduate School
Monterey, CA 93943-5000

5. Department of Operations Research (Code 55)......1
Naval Postgraduate School
Monterey, CA 93943-5000

6. Prof. Donald Gaver, Code OR-Gv......15
Naval Postgraduate School
Monterey, CA 93943-5000

7. Prof. Patricia Jacobs......15
Code OR/Jc
Naval Postgraduate School
Monterey, CA 93943-5000

8. Center for Naval Analyses......1
4401 Ford Avenue
Alexandria, VA 22302-0268

9. Dr. David Brillinger......1
Statistics Department
University of California
Berkeley, CA 94720

10. Dr. R. Gnanadesikan 1
Bellcore
435 South Street
Morris Township NJ 07960

11. Prof. Bernard Harris 1
Dept. of Statistics
University of Wisconsin
610 Walnut Street
Madison, WI 53706

12. Prof. W. M. Hinich 1
University of Texas
Austin, TX 78712

13. Prof. I. R. Savage..1
Dept. of Statistics
Yale University
New Haven, CT 06520

14. Prof. W. R. Schucany..1
Dept. of Statistics
Southern Methodist University
Dallas, TX 75222

15. Prof. D. C. Siegmund..1
Dept. of Statistics
Sequoia Hall
Stanford University
Stanford, CA 94305

16. Prof. H. Solomon 1
Department of Statistics
Sequoia Hall
Stanford University
Stanford, CA 94305

17. Dr. Ed Wegman..1
George Mason University
Fairfax, VA 22030

18. Dr. P. Welch 1
IBM Research Laboratory
Yorktown Heights, NY 10598

s

19. Dr. Neil Gerr 1
Office of Naval Research
Arlington, VA 22217

20. Prof. Roy Welsch...1
Sloan School
M.I.T.
Cambridge, MA 02139

21. Dr. J. Abrahams 1
Code 1111, Room 607
Mathematical Sciences Division
Office of Naval Research
800 North Quincy Street
Arlington, VA 22217-5000

22. Prof. J. R. Thompson...1
Dept. of Mathematical Science
Rice University
Houston, TX 77001

23. Dr. P. Heidelberger..1
IBM Research Laboratory
Yorktown Heights
New York, NY 10598

24. Prof. M. Leadbetter..1
Department of Statistics
University of North Carolina
Chapel Hill, NC 27514

25. Prof. D. L. Iglehart 1
Dept. of Operations Research
Stanford University
Stanford, CA 94350

26. Prof. J. B. Kadane..1
Dept. of Statistics
Carnegie-Mellon University
Pittsburgh, PA 15213

27. Prof. J. Lehoczky 1
Department of Statistics
Carnegie-Mellon University
Pittsburgh, PA 15213

28. Dr. J. Maar (R513) 1
National Security Agency
Fort Meade, MD 20755-6000

29. Prof. M. Mazumdar 1
Dept. of Industrial Engineering
University of Pittsburgh
Pittsburgh, PA 15235

30. Prof. M. Rosenblatt 1
Department of Mathematics
University of California, San Diego
La Jolla, CA 92093

31. Prof. H. Chernoff..1
Department of Statistics
Harvard University
1 Oxford Street
Cambridge, MA 02138

32. Dr. T. J. Ott 1
Bellcore
435 South Street
Morris Township, NJ 07960

33. Dr. Alan Weiss 1
AT&T Bell Laboratories
Mountain Avenue
Murray Hill, NJ 07974

34. Prof. Joseph R. Gani..1
Mathematics Department
University of California
Santa Barbara, CA 93106

35. Prof. Frank Samaniego 1
Statistics Department
University of California
Davis, CA 95616

s

36. Dr. James McKenna 1
AT&T Bell Laboratories
Mountain Avenue
Murray Hill, NJ 07974

37. Commander 1
Attn: G. F. Kramer
Naval Ocean Systems Center
Code 421
San Diego, CA 92152-5000

38. Prof. Tom A. Louis 1
School of Public Health
University of Minnesota
Mayo Bldg. A460
Minneapolis, MN 55455

39. Dr. Nan Laird 1
Biostatistics Dept.
Harvard School of Public Health
677 Huntington Ave.
Boston, MA 02115

40. Dr. Marvin Zelen 1
Biostatistics Department
Harvard School of Public Health
677 Huntington Ave.
Boston, MA 02115

41. Dr. John Orav 1
Biostatistics Department
Harvard School of Public Health
677 Huntington Ave.
Boston, MA 02115

42. Prof. R. Douglas Martin 1
Department of Statistics, GN-22
University of Washington
Seattle, WA 98195

43. Prof. W. Stuetzle 1
Department of Statistics
University of Washington
Seattle, WA 98195

44. Prof. F. W. Mosteller 1
Department of Statistics
Harvard University
1 Oxford St.
Cambridge, MA 02138

45. Dr. D. C. Hoaglin 1
Department of Statistics
Hardvard University
1 Oxford Street
Cambridge, MA 02138

46. Prof. N. D. Singpurwalla 1
George Washington University
Washington, DC 20052

47. Center for Naval Analysis 1
2000 Beauregard Street
Alexandria, VA 22311

48. Prof. George S. Fishman...1
Curr. in OR & Systems Analysis
University of North Carolina
Chapel Hill, NC 20742

49. Dr. Alan F. Petty 1
Code 7930
Naval Research Laboratory
Washington, DC 20375

50. Prof. Bradley Efron..1
Statistics Dept.
Sequoia Hall
Stanford University
Stanford, CA 94305

s

51. Prof. Carl N. Morris...1
Statistics Department
Harvard University
1 Oxford St.
Cambridge, MA 02138

52. Dr. John E. Rolph...1
RAND Corporation
1700 Main St.
Santa Monica, CA 90406

53. Prof. Linda V. Green..1
Graduate School of Business
Columbia University
New York, NY 10027

54. Dr. David Burman...1
AT&T Beli Telephone Laboratories
Mountain Avenue
Murray Hill, NJ 07974

55. Dr. Ed Coffman...1
AT&T Bell Telephone Laboratories
Mountain Avenue
Murray Hill, NJ 07974

56. Prof. William Jewell...1
Operations Research Department
University of California, Berkeley
Berkeley, CA 94720

57. Prof. D. C. Siegmund...1
Dept. of Statistics
Sequoia Hall
Stanford University
Stanford, CA 94305

58. Operations Research Center, Rm E40-164...1
Massachusetts Institute of Technology
Attn: R. C. Larson and J. F. Shapiro
Cambridge, MA 02139

59. Arthur P. Hurter, Jr. 1
Professor and Chairman
Dept. of Industrial Engineering and Management Sciences
Northwestern University
Evanston, IL 60201-9990

60. Institute for Defense Analysis 1
1800 North Beauregard
Alexandria, VA 22311

61. Prof. J. W. Tukey 1
Statistics Dept., Fine Hall
Princeton University
Princeton, NJ 08540

62. Dr. Daniel H. Wagner 1
Station Square One
Paoli, PA 19301

63. Dr. Colin Mallows..1
AT&T Bell Telephone Laboratories
Mountain Avenue
Murray Hill, NJ 07974

64. Dr. D. Pregibon..1
AT&T Bell Telephone Laboratories
Mountain Avenue
Murray Hill, NJ 07974

65. Dr. Jon Kettenring..1
Bellcore
435 South Street
Morris Township, NJ 07960

66. Prof. David L. Wallace 1
Statistics Dept.
University of Chicago
5734 S. University Ave.
Chicago, IL 60637

67. Dr. S. R. Dalal...1
AT&T Bell Telephone Laboratories
Mountain Avenue
Murray Hill, NJ 07974

68. Dr. M. J. Fischer 1
Defense Communications Agency
1860 Wiehle Avenue
Reston, VA 22070

69. Dr. Prabha Kumar
Defense Communications Agency 1
1860 Wiehle Avenue
Reston, VA 22070

70. Dr. B. Doshi 1
AT&T Bell Laboratories
HO 3M-335
Holmdel, NJ 07733

71. Dr. D. M. Lucantoni 1
AT&T Bell Laboratories
Holdmel, NJ 07733

72. Dr. V. Ramaswami 1
MRE 2Q-358
Bell Communications Research, Inc.
435 South Street
Morristown, NJ 07960

73. Prof. G. Shantikumar 1
The Management Science Group
School of Business Administration
University of California
Berkeley, CA 94720

74. Dr. D. F. Daley 1
Statistic Dept. (I.A.S.)
Australian National University
Canberra, A.C.T. 2606
AUSTRALIA

75. Dr. Guy Fayolle 1
I.N.R.I.A.
Dom de Voluceau-Rocquencourt
78150 Le Chesnay Cedex
FRANCE

76. Professor H. G. Daellenbach..1
Department of Operations Research
University of Canterbury
Christchurch, NEW ZEALAND

77. Koh Peng Kong..1
OA Branch, DSO
Ministry of Defense
Blk 29 Middlesex Road
SINGAPORE 1024

78. Professor Sir David Cox..1
Nuffield College
Oxford, OXI INF
ENGLAND

79. Dr. A. J. Lawrence..1
Department of Mathematical Statistics
University of Birmingham
P. O. Box 363
Birmingham B15 2TT
ENGLAND

80. Dr. F. P. Kelly..1
Statistics Laboratory
16 Mill Lane
Cambridge
ENGLAND

81. Dr. R. J. Gibbens..1
Statistics Laboratory
16 Mill Lane
Cambridge
ENGLAND

82. Dr. John Copas..1
Dept. of Mathematical Statistics
University of Birmingham
P. O. Box 363
Birmingham B15 2TT
ENGLAND

83. Dr. D. Vere-Jones..1
Dept. of Mathematics
Victoria University of Wellington
P. O. Box 196
Wellington
NEW ZEALAND

84. Prof. Guy Latouche..1
University Libre Bruxelles
C. P. 212
Blvd. De Triomphe
B-1050 Bruxelles
BELGIUM